

Freely floating structures trapping time-harmonic water waves (revisited)

Nikolay Kuznetsov and Oleg Motygin

Laboratory for Mathematical Modelling of Wave Phenomena,
Institute for Problems in Mechanical Engineering, Russian Academy of Sciences,
V.O., Bol'shoy pr. 61, St. Petersburg 199178, Russian Federation
E-mail: nikolay.g.kuznetsov@gmail.com; o.v.motygin@gmail.com

Abstract

We study the coupled small-amplitude motion of the mechanical system consisting of infinitely deep water and a structure immersed in it. The former is bounded above by a free surface, whereas the latter is formed by an arbitrary finite number of surface-piercing bodies floating freely. The mathematical model of time-harmonic motion is a spectral problem in which the frequency of oscillations serves as the spectral parameter. It is proved that there exist axisymmetric structures consisting of $N \geq 2$ bodies; every structure has the following properties: (i) a time-harmonic wave mode is trapped by it; (ii) some of its bodies (may be none) are motionless, whereas the rest of the bodies (may be none) are heaving at the same frequency as water. The construction of these structures is based on a generalization of the semi-inverse procedure applied earlier for obtaining trapping bodies that are motionless although float freely.

1 Introduction

This paper deals with the coupled problem describing the time-harmonic motion of the mechanical system that consists of an inviscid, incompressible, heavy fluid (water) in which a partially immersed structure floats freely. The latter means that there are no external forces acting on it other than gravity (for example, due to constraints on its motion). It is assumed that water occupies (together with the immersed part of the structure) a half-space and the structure consists of a finite number $N \geq 2$ of bounded surface-piercing bodies. The water motion is supposed to be irrotational and the surface tension is neglected on the free surface of water; moreover, the motion of the system is supposed to be of small amplitude near equilibrium, which allows us to use a linear model. In our previous papers [10] and [11], we considered the case of a single body which floats freely in water of finite and infinite depth, respectively.

In his pioneering article [5], F. John developed the time-dependent model for a single freely floating body. The two-dimensional version of the coupling conditions proposed by him (in particular, the equations of body's motion) were presented in the convenient matrix form in [8]. In [17], a similar form was developed for the three-dimensional problem and here we use its generalisation to the case of multiple bodies.

⁰The authors were supported by the Presidium of the Russian Academy of Sciences (Programme 44Π: "Fundamental Scientific Search Studies of Perspectives for Development of the Arctic Region").

Assuming that the water motion is simple harmonic in time, we reduce the time-dependent problem to a coupled spectral problem with the frequency of oscillations playing the role of spectral parameter; it appears in the boundary conditions, the radiation condition at infinity as well as in the equations of motion of each body. Since the system's energy is proved to be finite, it is possible to reformulate this spectral problem as an operator equation in a Hilbert space (see [17], where this approach was developed for a similar problem in a channel symmetric about its centre-plane). However, such a formulation is superfluous for our purpose of constructing trapping structures and the corresponding trapped modes.

In our papers [10] and [11], various uniqueness theorems were proved for the time-harmonic problem for finite and infinite depth of water, respectively. In the first of these papers, the original proof of John [6] was essentially simplified. For proving the uniqueness theorem in [11], a new non-dimensional form of the problem was proposed; this allowed us to evaluate the lower bound for frequencies at which the uniqueness is guaranteed. The last paper also contains various examples of a single body with the following properties. It is motionless, floats freely and traps axisymmetric wave modes. Earlier, similar examples were obtained for the two-dimensional problem in [8].

One has to keep in mind that in the vast majority of published papers the questions of uniqueness and existence of solutions and of trapped modes are studied for the scattering and radiation problems under the assumption that an immersed body or several such bodies are fixed (see [9] and [12] for surveys, whereas the most recent uniqueness theorems can be found in [7] and [16]). Before 2010, the only rigorous result for the problem of a freely floating body was that of John [6], who proved a uniqueness theorem (unfortunately, without formulating the problem explicitly). No other rigorous results about this problem had been obtained until recently. However, after 2005 a number of authors considered the question of trapped modes at the heuristic level and various two-dimensional and axisymmetric trapping structures were proposed by virtue of numerical computations (see [13, 14, 3, 19, 20, 21, 4], which are listed in the chronological order). In most of these papers, a simplified model is treated; it deals with a freely floating body constrained to the heave motion only. The exception is the article [21] in which an example of trapping structure is considered whose motion is combined (heave and sway).

In the present paper, our aim is to construct explicitly trapped modes, that is, eigensolutions of the coupled time-harmonic problem involving freely floating structures that consist of multiple bodies; some of these (may be none) are motionless, whereas the rest part of bodies (may be none) are in the heave motion. For this purpose we apply the so-called inverse method that replaces finding a solution to a problem in a given domain by determining a physically acceptable water region for a given solution. It is worth mentioning that this method widely used in continuum mechanics prior to the advent of computers (see [18] for a survey) appears in two forms distinguished by the involvement of boundary conditions. If some of these conditions, but not all, are prescribed at the outset, the method is referred to as semi-inverse and this particular form of it is used here.

The paper's plan is as follows. We begin with formulating the time-dependent problem in §2.1. Then we apply an ansatz that introduces complex-valued unknowns appropriate for considering time-harmonic oscillations and reduces the time-

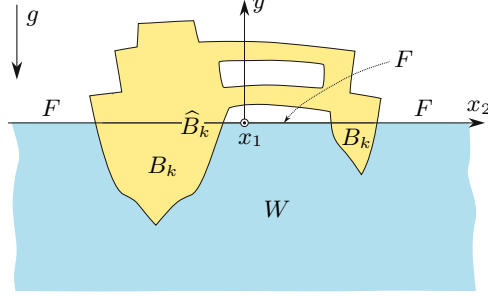


Figure 1: A body with two immersed parts.

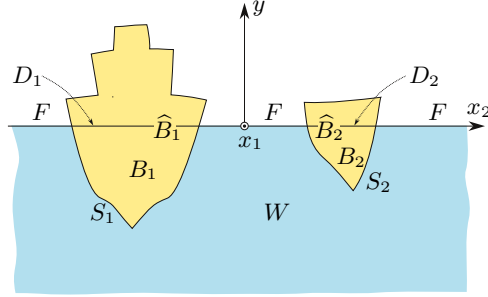


Figure 2: Definition sketch for two bodies.

dependent problem to a coupled spectral problem (§ 2.2). The brief § 3 deals with the energy of the coupled time-harmonic motion; it also contains the definition of a trapped mode. In the main § 4, various trapped modes and the corresponding axisymmetric water domains are constructed.

2 Statement of the problem

Let the Cartesian coordinates (\mathbf{x}, y) , $\mathbf{x} = (x_1, x_2)$, be such that the y -axis is directed upwards, whereas the mean free surface of water lies in the \mathbf{x} -plane, and so the water domain is a subset of $\mathbb{R}_-^3 = \{\mathbf{x} \in \mathbb{R}^2, y < 0\}$. The domain occupied by the k th body in its equilibrium position we denote by \hat{B}_k , $k = 1, \dots, N$; its immersed part $B_k = \hat{B}_k \cap \mathbb{R}_-^3 \neq \emptyset$ can consist of several connected components (see fig. 1). Let $B = \cup_{k=1}^N B_k$ and $W = \mathbb{R}_-^3 \setminus \overline{B}$ denote the structure's submerged part and the water domain, respectively. It is supposed that W is simply connected, whereas B has at least $N \geq 2$ connected components (see fig. 2); their number is greater than N if bodies like that shown in fig. 1 are present. Furthermore, $S_k = \partial B_k \cap \mathbb{R}_-^3$ and $F = \{y = 0\} \setminus (\cup_{k=1}^N \overline{D_k})$ stand for the wetted surface of the k th body and the free surface of water in its mean position, respectively; here $D_k = \hat{B}_k \cap \{y = 0\}$ (see fig. 2); for the sake of brevity we put $S = \cup_{k=1}^N S_k$.

2.1 Time-dependent problem

In the linearised time-dependent setting for $N = 1$ obtained in [5], the coupled motion is described in terms of the so-called first-order unknowns: the velocity potential for the water motion and a vector characterising the motion of the body's centre of mass.

Here we give the problem's statement for $N \geq 2$ on the basis of the assumptions listed above which are the same as in [5].

The first-order velocity potential $\Phi(\mathbf{x}, y; t)$ exists because the water motion is supposed to be irrotational and W is simply connected. Hence the velocity field is equal to $\nabla\Phi(\mathbf{x}, y; t)$ ($\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_y)$ is the spacial gradient), and the continuity equation takes the form:

$$\nabla^2\Phi = 0 \quad \text{in } W \quad \text{for all } t. \quad (1)$$

In order to specify the behaviour of Φ near ∂W , we require that this function belongs to the Sobolev class $H_{\text{loc}}^1(W)$. The standard linear boundary condition on the free surface has the following form:

$$\partial_{tt}\Phi + g\partial_y\Phi = 0 \quad \text{on } F \quad \text{for all } t. \quad (2)$$

Here $g > 0$ is the acceleration due to gravity that acts in the direction opposite to the y -axis (see fig. 1). Equality (2) is a consequence of Bernoulli's equation and the kinematic condition both taken linearised on the mean free surface; the first of these conditions expresses the fact that the pressure is constant on F , whilst the second one means that there is no transfer of matter across F .

Furthermore, the following set of vectors $\mathbf{q}^{(k)}(t) \in \mathbb{R}^6$ characterises the motion of the centre of mass of the k th body about its given equilibrium position $(\mathbf{x}_0^{(k)}, y_0^{(k)})$; namely, for every $k = 1, \dots, N$

- the horizontal and vertical displacements are $q_1^{(k)}$, $q_2^{(k)}$ and $q_4^{(k)}$, respectively;
- $q_3^{(k)}$ and $q_5^{(k)}$, $q_6^{(k)}$ are the angles of rotation about the axes that go through $(\mathbf{x}_0^{(k)}, y_0^{(k)})$ parallel to the y - and x_1 -, x_2 -axes, respectively.

The following kinematic condition couples Φ and $\mathbf{q}^{(k)}$ for each body:

$$\partial_{\mathbf{n}}\Phi(\mathbf{x}, y; t) = [\mathbf{n}(\mathbf{x}, y)]^\top \mathbf{D}_0^{(k)}(\mathbf{x}, y) \dot{\mathbf{q}}^{(k)}(t) \quad \text{on } S_k \quad \text{for all } t. \quad (3)$$

By $^\top$ the operation of matrix transposition is denoted (a vector is considered as a one-column matrix) and \mathbf{n} is the unit normal to ∂W (in particular, to S_k) directed to the exterior of W . The vector $\dot{\mathbf{q}}$ (the dot stands for the time derivative) characterises the body motion in the following manner: $(\dot{q}_1, \dot{q}_2, \dot{q}_4)^\top$ is the velocity vector of the translational motion and $(\dot{q}_3, \dot{q}_5, \dot{q}_6)^\top$ is the vector of angular velocities. The 3×6 matrix $\mathbf{D}_0^{(k)}(\mathbf{x}, y)$ is defined as follows:

$$\mathbf{D}_0^{(k)}(\mathbf{x}, y) = \mathbf{D}(\mathbf{x} - \mathbf{x}_0^{(k)}, y - y_0^{(k)}), \quad \text{where } \mathbf{D}(\mathbf{x}, y) = \begin{bmatrix} 1 & 0 & x_2 & 0 & 0 & -y \\ 0 & 1 & -x_1 & 0 & y & 0 \\ 0 & 0 & 0 & 1 & -x_2 & x_1 \end{bmatrix}.$$

The latter describes the motion of a rigid body so that its elements are in conformity with the order of components of the corresponding vector \mathbf{q} .

The linearised system of equations describing the motion of the k th body expresses the conservation of its linear and angular momentum. In the absence of external forces other than gravity this system is as follows:

$$\mathbf{E}_0^{(k)} \ddot{\mathbf{q}}^{(k)}(t) = - \int_{S_k} \partial_t \Phi(\mathbf{x}, y; t) [\mathbf{D}_0^{(k)}(\mathbf{x}, y)]^\top \mathbf{n}(\mathbf{x}, y) \, ds - g \mathbf{K}_0^{(k)} \mathbf{q}^{(k)}(t) \quad \text{for all } t. \quad (4)$$

Here $\ddot{\mathbf{q}}^{(k)}(t)$ is the acceleration vector of the k th body and $\mathbf{E}_0^{(k)}$ is its mass/inertia matrix defined as follows:

$$\mathbf{E}_0^{(k)} = \rho_0^{-1} \int_{\hat{B}_k} \rho_k(\mathbf{x}, y) [\mathbf{D}_0^{(k)}(\mathbf{x}, y)]^\top \mathbf{D}_0^{(k)}(\mathbf{x}, y) d\mathbf{x} dy,$$

where $\rho_k(\mathbf{x}, y) \geq 0$ is the density distribution within the k th body and $\rho_0 > 0$ is the constant density of water. A direct calculation gives the following explicit form of this matrix:

$$\mathbf{E}_0^{(k)} = \begin{pmatrix} I^{\hat{B}_k} & 0 & 0 & 0 & 0 & 0 \\ 0 & I^{\hat{B}_k} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{x_1 x_1}^{\hat{B}_k} + I_{x_2 x_2}^{\hat{B}_k} & 0 & -I_{x_1 y}^{\hat{B}_k} & -I_{x_2 y}^{\hat{B}_k} \\ 0 & 0 & 0 & I^{\hat{B}_k} & 0 & 0 \\ 0 & 0 & -I_{x_1 y}^{\hat{B}_k} & 0 & I_{x_2 x_2}^{\hat{B}_k} + I_{yy}^{\hat{B}_k} & -I_{x_1 x_2}^{\hat{B}_k} \\ 0 & 0 & -I_{x_2 y}^{\hat{B}_k} & 0 & -I_{x_1 x_2}^{\hat{B}_k} & I_{x_1 x_1}^{\hat{B}_k} + I_{yy}^{\hat{B}_k} \end{pmatrix}. \quad (5)$$

The matrix elements are composed of various moments of the whole body \hat{B}_k ; namely,

$$I^{\hat{B}_k} = \rho_0^{-1} \int_{\hat{B}_k} \rho_k(\mathbf{x}, y) d\mathbf{x} dy, \quad I_{\sigma\tau}^{\hat{B}_k} = \rho_0^{-1} \int_{\hat{B}_k} \rho_k(\mathbf{x}, y) (\sigma - \sigma_0^{(k)}) (\tau - \tau_0^{(k)}) d\mathbf{x} dy.$$

Here σ and τ stand for the corresponding coordinates, whereas $\sigma_0^{(k)}$ and $\tau_0^{(k)}$ are taken equal to $x_{10}^{(k)}, x_{20}^{(k)}$ and $y_0^{(k)}$ so that these coordinates are the same as σ and τ , respectively. In formula (5), it is taken into account that for any coordinate σ we have

$$I_{\sigma}^{\hat{B}_k} = \rho_0^{-1} \int_{\hat{B}_k} \rho_k(\mathbf{x}, y) (\sigma - \sigma_0^{(k)}) d\mathbf{x} dy = 0,$$

which is a consequence of the definition of $(\mathbf{x}_0^{(k)}, y_0^{(k)})$. It is obvious that every matrix $\mathbf{E}_0^{(k)}$ is symmetric; moreover, it is straightforward to verify that all these matrices are positive definite.

In the right-hand side of (4), we have forces and their moments; namely, the first term is of the hydrodynamic origin, whereas the second one is related to the buoyancy (see, for example, [5] and [15]). The matrix in the second term has the following blockwise form:

$$\mathbf{K}_0^{(k)} = \begin{pmatrix} \mathbb{O}_3 & \mathbb{O}_3 \\ \mathbb{O}_3 & \widehat{\mathbf{K}}_0^{(k)} \end{pmatrix}, \quad \text{where } \widehat{\mathbf{K}}_0^{(k)} = \begin{pmatrix} I^{D_k} & I_2^{D_k} & -I_1^{D_k} \\ I_2^{D_k} & I_{22}^{D_k} + I_y^{B_k} & -I_{12}^{D_k} \\ -I_1^{D_k} & -I_{12}^{D_k} & I_{11}^{D_k} + I_y^{B_k} \end{pmatrix} \quad (6)$$

and \mathbb{O}_3 is the null 3×3 matrix. The elements of $\widehat{\mathbf{K}}_0^{(k)}$ involve the following moments:

$$I^{D_k} = \int_{D_k} d\mathbf{x}, \quad I_y^{B_k} = \int_{B_k} (y - y_0^{(k)}) d\mathbf{x} dy, \\ I_i^{D_k} = \int_{D_k} (x_i - x_{i0}^{(k)}) d\mathbf{x}, \quad I_{ij}^{D_k} = \int_{D_k} (x_i - x_{i0}^{(k)}) (x_j - x_{j0}^{(k)}) d\mathbf{x}, \quad i, j = 1, 2.$$

It is clear that the matrix $\mathbf{K}_0^{(k)}$ is symmetric.

Relations (1)–(4) must be augmented by the following subsidiary conditions guaranteeing equilibrium for each of N floating bodies and its stability:

- $I^{\widehat{B}_k} = \int_{B_k} d\mathbf{x} dy$; that is, the mass of water displaced by the k th body is equal to its own mass (Archimedes' law).
- $\int_{B_k} (x_i - x_{i0}^{(k)}) d\mathbf{x} dy = 0$, $i = 1, 2$; that is, the center of buoyancy of the k th body lies on the same vertical line as its centre of mass (see, for example, [15, § 8.2.3]).
- Every matrix $\mathbf{K}_0^{(k)}$ is positive semi-definite, whereas $\widehat{\mathbf{K}}_0^{(k)}$ is positive definite. This is the classical condition yielding the stability of the equilibrium position of the k th body (see, for example, [5, § 2.4]). As usual, the stability means that an instantaneous, infinitesimal disturbance causes the position changes remaining infinitesimal for all subsequent times with the exception of purely horizontal motion.

Of course, relations (1)–(4) must be complemented by proper initial conditions in order to obtain a well-posed initial-value problem (see [2], § 3, where this question is considered). However, our aim is to study free time-harmonic oscillations of the system not depending on the initial conditions. A heuristic explanation how this phenomenon is to be conceived is discussed in detail in [6], p. 46.

2.2 Time-harmonic problem

In order to formulate the problem of coupled time-harmonic motion for the mechanical system described in § 2.1, we assume $\omega > 0$ to be the radian frequency of oscillations and represent the velocity potential and the displacement vectors in the following form:

$$(\Phi(\mathbf{x}, y; t), \mathbf{q}^{(1)}(t), \dots, \mathbf{q}^{(N)}(t)) = \Re\{e^{-i\omega t}(\varphi(\mathbf{x}, y), i\boldsymbol{\chi}^{(1)}, \dots, i\boldsymbol{\chi}^{(N)})\}.$$

Here φ is a complex-valued function and $\boldsymbol{\chi}^{(k)} \in \mathbb{C}^6$. Substituting the latter expression into relations (1)–(4), we immediately get

$$\nabla^2 \varphi = 0 \quad \text{in } W, \tag{7}$$

$$\partial_y \varphi - \nu \varphi = 0 \quad \text{on } F, \quad \nu = \frac{\omega^2}{g}, \tag{8}$$

$$\partial_{\mathbf{n}} \varphi = \omega \mathbf{n}^T \mathbf{D}_0^{(k)} \boldsymbol{\chi}^{(k)} \quad \text{on } S_k, \tag{9}$$

$$\omega^2 \mathbf{E}_0^{(k)} \boldsymbol{\chi}^{(k)} = -\omega \int_{S_k} \varphi [\mathbf{D}_0^{(k)}(\mathbf{x}, y)]^T \mathbf{n} ds + g \mathbf{K}_0^{(k)} \boldsymbol{\chi}^{(k)}, \tag{10}$$

where the last two conditions must hold for every $k = 1, \dots, N$. We specify the behaviour of φ at infinity so that the velocity potential Φ describes outgoing waves by imposing the radiation condition

$$\int_{W \cap \{|\mathbf{x}|=a\}} |\partial_{|\mathbf{x}|} \varphi - i\nu \varphi|^2 ds = o(1) \quad \text{as } a \rightarrow \infty. \tag{11}$$

It is natural that this condition is the same as in the water-wave problem for fixed obstacles (see, for example, [6]). In the boundary value problem (7)–(11), ω is a spectral parameter sought together with the eigenvector $(\varphi, \boldsymbol{\chi}^{(1)}, \dots, \boldsymbol{\chi}^{(N)})$.

Since, generally speaking, ∂W is not smooth and we assumed that $\varphi \in H_{\text{loc}}^1(W)$, it is sufficient to understand the whole set of relations (7)–(9) in the sense of the

integral identity

$$\int_W \nabla \varphi \nabla \psi \, d\mathbf{x} \, dy = \nu \int_F \varphi \psi \, d\mathbf{x} + \omega \sum_{k=1}^N \int_{S_k} \psi \mathbf{n}^\top \mathbf{D}_0^{(k)} \boldsymbol{\chi}^{(k)} \, ds, \quad (12)$$

which must hold for all smooth functions ψ having a compact support in \overline{W} .

3 On the energy of the coupled time-harmonic motion

In [10] and [11], the following intuitively ‘obvious’ assertion was proved for a single body immersed in water of finite and infinite depth, respectively. If there is no net input of energy into the motion of water or bodies forming the structure, then there is no wave radiation to infinity, and so the total energy of the coupled motion is finite. In this section, we generalise this fact to the case of a structure consisting of multiple bodies, thus showing that a non-trivial solution of problem (7)–(10) and (11) describes waves trapped by the structure.

It is known (see, for example, [9, §2.2.1]) that if φ satisfies relations (7), (8) and (11), then it has the following asymptotic representation at infinity:

$$\varphi(\mathbf{x}, y) = A(\theta) |\mathbf{x}|^{-1/2} e^{\nu(y + i|\mathbf{x}|)} + R(\mathbf{x}, y). \quad (13)$$

Moreover, the remainder behaves as follows:

$$|R|, |\nabla R| = O\left(e^{\nu y} (1 + \nu |\mathbf{x}|)^{-3/2} + [\nu^2 (|\mathbf{x}|^2 + y^2)]^{-1}\right) \text{ as } \nu^2 (|\mathbf{x}|^2 + y^2) \rightarrow \infty, \quad (14)$$

and the equality

$$\frac{1}{2} \int_0^{2\pi} |A(\theta)|^2 \, d\theta = -\Im \int_S \overline{\varphi} \partial_{\mathbf{n}} \varphi \, ds \quad (15)$$

holds for the coefficient in the leading term. Here θ is the polar angle in the (x_1, x_2) -plane measured anti-clockwise and we recall that S denotes the union of all S_k .

In the same way as in [10, 11] it follows from (9), (10) and (15) that

$$\frac{1}{2} \int_0^{2\pi} |A(\theta)|^2 \, d\theta = \Im \sum_{k=1}^N \left\{ \omega^2 \overline{\boldsymbol{\chi}^{(k)}}^\top \mathbf{E}_0^{(k)} \boldsymbol{\chi}^{(k)} - g \overline{\boldsymbol{\chi}^{(k)}}^\top \mathbf{K}_0^{(k)} \boldsymbol{\chi}^{(k)} \right\}.$$

Since the expression in braces is real, we get that $A(\theta) = 0$, and so the behaviour of φ at infinity is given by formula (14). Combining this fact and the assumption that $\varphi \in H_{\text{loc}}^1(W)$, we arrive at the following assertion.

LEMMA 3.1. *Let $(\varphi, \boldsymbol{\chi}^{(1)}, \dots, \boldsymbol{\chi}^{(N)})$ be a solution of problem (7)–(10) and (11), then the first component φ belongs to the Sobolev space $H^1(W)$.*

Now we are in a position to extend Proposition 1 in [11] to the case of structures consisting of multiple bodies.

PROPOSITION 3.2. *Let $(\varphi, \boldsymbol{\chi}^{(1)}, \dots, \boldsymbol{\chi}^{(N)})$ be a solution of problem (7)–(11), then*

$$\int_W |\nabla \varphi|^2 \, d\mathbf{x} \, dy < \infty \quad \text{and} \quad \int_F |\varphi|^2 \, d\mathbf{x} < \infty, \quad (16)$$

that is, the kinetic and potential energy of the water motion are finite. Moreover, the following equality holds

$$\int_W |\nabla \varphi|^2 d\mathbf{x} dy + \omega^2 \sum_{k=1}^N \overline{\chi^{(k)}}^\top \mathbf{E}_0^{(k)} \chi^{(k)} = \nu \int_F |\varphi|^2 d\mathbf{x} + g \overline{\chi^{(k)}}^\top \mathbf{K}_0^{(k)} \chi^{(k)}, \quad (17)$$

thus expressing the equipartition of energy of the coupled motion.

Proof. Relations (16) are an immediate consequence of Lemma 3.1. For proving equality (17) we introduce an infinitely differentiable cut-off function $\zeta_a(|\mathbf{x}|, y)$ equal to one on $\{|\mathbf{x}| \leq a, -a \leq y \leq 0\}$ and to zero when either $|\mathbf{x}| \geq a+1$ or $y \leq -a-1$. Let a be so large that the wetted surface S of the whole structure lies within the truncated cylinder $\{|\mathbf{x}| < a, y > -a\}$, then substituting $\psi = \overline{\varphi} \zeta_a(|\mathbf{x}|)$ into (12), we see that relations (13) allow us to let $a \rightarrow \infty$, which combined with (10) gives the required equality.

These assertions show that if $(\varphi, \chi^{(1)}, \dots, \chi^{(N)})$ is a solution of problem (7)–(11) with complex-valued components, then its real and imaginary parts separately satisfy this problem. This allows us to consider $(\varphi, \chi^{(1)}, \dots, \chi^{(N)})$ as an element of the real product space $H^1(W) \times \mathbb{R}^{6N}$ in what follows.

DEFINITION 3.3. Let the subsidiary conditions concerning the equilibrium position (see § 2) hold for a freely floating structure. A non-trivial real solution

$$(\varphi, \chi^{(1)}, \dots, \chi^{(N)}) \in H^1(W) \times \mathbb{R}^{6N}$$

of problem (7)–(10) is called a *mode trapped* by the structure, whereas the corresponding value of ω is referred to as a *trapping frequency*.

4 Trapped modes and the corresponding axisymmetric trapping structures

In this section, we construct trapped modes with axisymmetric velocity fields and the corresponding structures consisting of arbitrarily large but fixed number N of axisymmetric bodies; a part of these (may be all or none) are motionless, whereas the rest (may be none or all) are heaving. In order to find such bodies we modify the semi-inverse procedure applied in [10] using not only a special choice of the velocity potential, but also a particular form of the vectors $\chi^{(k)}$, $k = 1, \dots, N$. The potential is defined so that it satisfies the Laplace equation and the free-surface boundary condition; moreover, it does not radiate waves to infinity. Since a set of bodies with axisymmetric immersed parts is sought, level surfaces of a Stokes stream function are used for finding admissible wetted surfaces of motionless bodies. Besides, a special term is added to this stream function in order to find admissible wetted surfaces of heaving bodies as level surfaces of the modified stream function (another its modification was proposed in [14]).

4.1 Velocity potentials of trapped modes

Let us fix $\omega > 0$ arbitrarily, and this value will serve as the trapping frequency. A trapped mode is sought in the form $(\omega \nu^{-2} \varphi_*, d\chi_*^{(1)}, \dots, d\chi_*^{(N)})$ with dimensionless

φ_* and $\chi_*^{(k)}$ and the diagonal matrix $\text{diag}\{\nu^{-1}, \nu^{-1}, 1, \nu^{-1}, 1, 1\}$ as \mathbf{d} . For the sake of brevity, $(\varphi_*, \chi_*^{(1)}, \dots, \chi_*^{(N)})$ will also be referred to as trapped mode.

Following the semi-inverse method, we define φ_* explicitly for $y \leq 0$ (cf. [11], § 3.1):

$$\begin{aligned} \varphi_*(\nu|\mathbf{x}|, \nu y) = & 2 \int_0^\infty (k \cos k\nu y + \sin k\nu y) I_0(k\nu|\mathbf{x}|) K_1(k\nu r) \frac{k^2 dk}{k^2 + 1} \\ & - \pi^2 e^{\nu y} J_0(\nu|\mathbf{x}|) Y_1(\nu r) \quad \text{when } |\mathbf{x}| < r, \end{aligned} \quad (18)$$

$$\begin{aligned} \varphi_*(\nu|\mathbf{x}|, \nu y) = & 2 \int_0^\infty (k \cos k\nu y + \sin k\nu y) K_0(k\nu|\mathbf{x}|) I_1(k\nu r) \frac{k^2 dk}{k^2 + 1} \\ & - \pi^2 e^{\nu y} Y_0(\nu|\mathbf{x}|) J_1(\nu r) \quad \text{when } |\mathbf{x}| > r. \end{aligned} \quad (19)$$

Here r is a length to be specified below, Y_0 and Y_1 are the Neumann functions of order zero and one, respectively, whereas J_0 , J_1 and I_0 , I_1 , K_0 , K_1 denote the standard and modified Bessel functions of the corresponding orders.

Using the asymptotic behaviour of I_1 and K_1 at infinity (see formulae 9.7.1 and 9.7.2 in [1] or [23], §§ 7.23, 7.3), one immediately obtains that both integrals diverge when $|\mathbf{x}| = r$ and $y = 0$, and so φ_* has a singularity there. Furthermore, it is straightforward to verify that φ_* can be extended to the whole \mathbb{R}_-^3 so that the extension is harmonic. Moreover, the following boundary condition

$$\partial_y \varphi_* - \nu \varphi_* = 0 \quad \text{holds on } \partial \mathbb{R}_-^3 \setminus \{|\mathbf{x}| = r, y = 0\}. \quad (20)$$

Let us choose r so that $\nu r = j_{1,m}$, where $j_{1,m}$ is one of the positive zeros of J_1 . (These zeros form an infinite sequence arranged in ascending order as m increases.) According to this choice of r , the second term in the right-hand side of (19) (it describes an outgoing wave) vanishes. In what follows, we write r_m for $\nu^{-1} j_{1,m}$ and denote by φ_m the function φ_* with $r = r_m$. Therefore, $\varphi_m \in H^1(W)$ for any domain W obtained by removing some neighbourhood of the circumference $\{|\mathbf{x}| = r_m, y = 0\}$ from \mathbb{R}_-^3 . Indeed, φ_m satisfies the same estimates at infinity as the remainder R in (13). Thus, any function φ_m defined by formulae (18) and (19) with $r = r_m$ can serve as the first component of an eigensolution provided the vectors $\chi_*^{(k)}$ and the water domain W are chosen properly.

4.2 Stokes stream functions and streamlines

In order to construct water domains we begin with introducing Stokes stream functions that corresponds to the sequence φ_m . Namely, ψ_m is defined by virtue of the following relations:

$$\partial_{|\mathbf{x}|} \varphi_m = -(\nu|\mathbf{x}|)^{-1} \partial_y \psi_m, \quad \partial_y \varphi_m = (\nu|\mathbf{x}|)^{-1} \partial_{|\mathbf{x}|} \psi_m. \quad (21)$$

These equations give for $m = 1, 2, \dots$:

$$\begin{aligned} \psi_m(\nu|\mathbf{x}|, \nu y) = & -\pi^2 \nu|\mathbf{x}| e^{\nu y} J_1(\nu|\mathbf{x}|) Y_1(j_{1,m}) \\ & - 2 \nu|\mathbf{x}| \Psi(\nu|\mathbf{x}|, \nu r_m, \nu y) \quad \text{for } |\mathbf{x}| < r_m, y \leq 0, \end{aligned} \quad (22)$$

$$\psi_m(\nu|\mathbf{x}|, \nu y) = -2 \nu|\mathbf{x}| \Psi(\nu r_m, \nu|\mathbf{x}|, \nu y) \quad \text{for } |\mathbf{x}| > r_m, y \leq 0,$$

$$\text{where } \Psi(\sigma, \tau, \eta) = \int_0^\infty (k \sin k\eta - \cos k\eta) I_1(k\sigma) K_1(k\tau) \frac{k^2 dk}{k^2 + 1}.$$

The last function is defined for (σ, τ, η) such that $\eta \leq 0$, $0 \leq \sigma \leq \tau$ and $\eta \neq 0$ when $\sigma = \tau$. The constant of integration in this definition of ψ_m is chosen so that $\psi_m(\nu|\mathbf{x}|, \nu y) \rightarrow 0$ as $\nu^2[(|\mathbf{x}| - r_m)^2 + y^2] \rightarrow \infty$.

Since the velocity field is axisymmetric, by a streamline we mean the curve in the $(\nu|\mathbf{x}|, \nu y)$ -plane given by the equation $\psi_m(\nu|\mathbf{x}|, \nu y) = v$ with a constant v . (In fact, this equation defines axisymmetric surfaces and streamlines are their vertical cross-sections.) First, we formulate some properties of streamlines that will be used below.

PROPOSITION 4.1. (i) *In $Q = \{\nu|\mathbf{x}| > 0, \nu y < 0\}$, streamlines are smooth curves; their end-points belong to ∂Q for $v \neq 0$ and to $\partial Q \cup \{\infty\}$ for $v = 0$.*

(ii) *A streamline emanates from every point on the half-axis $\{\nu|\mathbf{x}| > 0, \nu y = 0\}$, except for the points, where $\psi_m(\nu|\mathbf{x}|, \nu y)$ attains its local extrema, and the point $(\nu r_m, 0)$.*

(iii) *For every $m \geq 1$ and all positive ν and v there exists a streamline such that $y = 0$ at both its ends and the point $(\nu r_m, 0)$ belongs to the segment connecting these end-points.*

This proposition is proved in [11], pp. 150–152.

4.3 Motionless trapping structures

It follows from (21) that $\partial_{\mathbf{n}}\varphi_m$ vanishes on every streamline of ψ_m , and so if $y = 0$ at both ends of a streamline, then the corresponding axisymmetric surface can serve as the wetted boundary of a motionless body or as a part of such boundary (see fig. 3, where the first of these options is realised for the body floating underneath the other one, whereas the latter body realises the second option). Assertion (iii) of the last proposition implies that one of the structure's motionless bodies, say, B_N can be taken so that φ_m is harmonic in W ; that is, the singularity of φ_m is separated from the water domain by the wetted surface S_N coinciding with a streamline surrounding the singularity. The next step is to prove the following assertion.

PROPOSITION 4.2. *For any given positive integer M there exists $m_*(M)$ such that the function $\psi_m(\nu|\mathbf{x}|, \nu y)$ has $M - 1$ local extrema on $\{\nu|\mathbf{x}| \in (0, j_{1,M}), \nu y = 0\}$ provided $m \geq m_*(M)$. The points, where these extrema are attained, tend to $(j_{0,l}, 0)$ as $m \rightarrow \infty$; here $j_{0,l}$, $l = 1, \dots, M$, are zeros of the Bessel function J_0 .*

Proof. Since extrema of ψ_m and $c\psi_m$, where c is a non-zero constant, are attained at the same points, we consider

$$\frac{\psi_m(\nu|\mathbf{x}|, \nu y)}{Y_1(j_{1,m})} = -\pi^2 \nu|\mathbf{x}| e^{\nu y} J_1(\nu|\mathbf{x}|) + \Lambda(\nu|\mathbf{x}|, \nu y), \quad (23)$$

$$\text{where } \Lambda(\nu|\mathbf{x}|, \nu y) = \frac{2\nu|\mathbf{x}|}{Y_1(j_{1,m})} \int_0^\infty [k \sin(k\nu y) - \cos(k\nu y)] I_1(k\nu|\mathbf{x}|) K_1(k j_{1,m}) \frac{k^2 dk}{k^2 + 1}$$

according to (22). Let us show that $\Lambda(\nu|\mathbf{x}|, 0)$ and $\nabla_{|\mathbf{x}|, y} \Lambda(\nu|\mathbf{x}|, \nu y)|_{y=0}$ tend to zero as $m \rightarrow \infty$, uniformly for $\nu|\mathbf{x}| \in [0, j_{1,M}]$.

We have that

$$|\Lambda(\nu|\mathbf{x}|, 0)| \leq \frac{\nu|\mathbf{x}|}{|Y_1(j_{1,m})|} \int_0^\infty k I_1(k\nu|\mathbf{x}|) K_1(k j_{1,m}) dk$$

because I_1 and K_1 are positive functions. Moreover, the right-hand side is equal to

$$\frac{(\nu|\mathbf{x}|)^2}{j_{1,m}|Y_1(j_{1,m})|[j_{1,m}^2 - (\nu|\mathbf{x}|)^2]} \quad \text{for } m \geq M+1 \text{ and } \nu|\mathbf{x}| \in [0, j_{1,M}], \quad (24)$$

which is a consequence of formula 1.12.4.2 in [22]:

$$\int_0^\infty k I_\mu(ak) K_\mu(bk) dk = \frac{(ab^{-1})^\mu}{b^2 - a^2} \quad \text{for } \mu > -1. \quad (25)$$

Note that formulae 9.5.12 and 9.2.2 in [1] give

$$j_{1,m} = \pi \left(m + \frac{1}{4} \right) + O(m^{-1}) \text{ and } Y_1(j_{1,m}) = (-1)^{m+1} \sqrt{\frac{2}{\pi^2 m}} + O(m^{-3/2}) \text{ as } m \rightarrow \infty. \quad (26)$$

Since M is fixed, (24) implies that

$$\max\{|\Lambda(\nu|\mathbf{x}|, 0)| : \nu|\mathbf{x}| \in [0, j_{1,M}]\} = O(m^{-5/2}) \text{ as } m \rightarrow \infty.$$

Using formula 9.6.28 in [1], we write

$$\partial_{|\mathbf{x}|} \Lambda(\nu|\mathbf{x}|, 0) = \frac{2\nu^2|\mathbf{x}|}{Y_1(j_{1,m})} \int_0^\infty I_0(k\nu|\mathbf{x}|) K_1(kj_{1,m}) \frac{k^3 dk}{k^2 + 1}.$$

Hence

$$\begin{aligned} |\partial_{|\mathbf{x}|} \Lambda(\nu|\mathbf{x}|, 0)| &\leq \frac{\nu^2|\mathbf{x}|}{|Y_1(j_{1,m})|} \int_0^\infty k^2 I_0(k\nu|\mathbf{x}|) K_1(kj_{1,m}) dk \\ &= \frac{2\nu^2|\mathbf{x}|j_{1,m}}{|Y_1(j_{1,m})|[j_{1,m}^2 - (\nu|\mathbf{x}|)^2]^2}. \end{aligned}$$

To get the last equality we differentiate (25), where $\mu = 0$, with respect to b and use the identity $K_0'(z) = -K_1(z)$. Now from (26) we obtain that

$$\max\{|\partial_{|\mathbf{x}|} \Lambda(\nu|\mathbf{x}|, 0)| : \nu|\mathbf{x}| \in [0, j_{1,M+1}]\} = O(m^{-5/2}) \text{ as } m \rightarrow \infty.$$

Furthermore, we have

$$\partial_y \Lambda(\nu|\mathbf{x}|, \nu y)|_{y=0} = \frac{2\nu^2|\mathbf{x}|}{Y_1(j_{1,m})} \int_0^\infty I_1(k\nu|\mathbf{x}|) K_1(kj_{1,m}) \frac{k^4 dk}{k^2 + 1}$$

and

$$\begin{aligned} |\partial_y \Lambda(\nu|\mathbf{x}|, \nu y)|_{y=0} &\leq \frac{\nu^2|\mathbf{x}|}{|Y_1(j_{1,m})|} \int_0^\infty k^3 I_1(k\nu|\mathbf{x}|) K_1(kj_{1,m}) dk \\ &= \frac{8\nu^3|\mathbf{x}|^2 j_{1,m}}{|Y_1(j_{1,m})|[j_{1,m}^2 - (\nu|\mathbf{x}|)^2]^3}. \end{aligned}$$

Again, the last equality is obtained by differentiation with respect to a and b of (25) with $\mu = 0$ and using that $K_0'(z) = -K_1(z)$ and $I_0'(z) = I_1(z)$. Then (26) yields that

$$\max\{|\partial_y \Lambda(\nu|\mathbf{x}|, \nu y)| : \nu|\mathbf{x}| \in [0, j_{1,M}], y = 0\} = O(m^{-9/2}) \text{ as } m \rightarrow \infty.$$

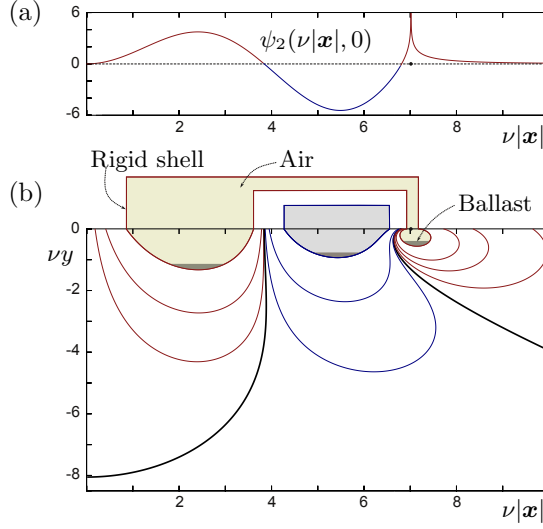


Figure 3: (a) The trace $\psi_2(\nu|\mathbf{x}|, 0)$. (b) Streamlines $\psi_2(\nu|\mathbf{x}|, \nu y) = v$ for various values of v ; nodal lines ($v = 0$) are bold. Straight segments show how wetted surfaces are connected above the free surface to form 2 bodies. Darkly shaded layers show the ballast guaranteeing that the bodies are in equilibrium.

It is easy to observe that the first term in the right-hand side of (23) considered in \overline{Q} attains its local extrema on the free surface. Since $(zJ_1(z))'_z = zJ_0(z)$ [1, 9.1.30], the extrema are attained at $(j_{0,l}, 0)$, $l = 1, 2, \dots$. The sign of these extrema alternates being strictly positive at maxima (negative at minima) and at these points the y -derivative is strictly positive (negative). It is clear that $(j_{0,l}, 0) \in \Upsilon = \{\nu|\mathbf{x}| \in (0, j_{1,M}), \nu y = 0\}$ for all $l = 1, 2, \dots, M - 1$.

In view of asymptotic estimates obtained for $\Lambda(\nu|\mathbf{x}|, 0)$, $\nabla_{|\mathbf{x}|, y} \Lambda(\nu|\mathbf{x}|, \nu y)|_{y=0}$ as $m \rightarrow \infty$ we see that the contribution of the second term in the right-hand side of (23) is negligible. Therefore, the behaviour of $\psi_m(\nu|\mathbf{x}|, \nu y)$ on Υ and in Q near Υ is the same as that of the first term in the right-hand side of (23). There are $M - 1$ points of extrema of $\psi_m(\nu|\mathbf{x}|, \nu y)$ on Υ (alternating in sign, strictly positive maximums and negative minimums), and the points of extrema approach $(j_{0,l}, 0)$ ($l = 1, 2, \dots, M - 1$) as $m \rightarrow \infty$.

A consequence of Propositions 4.1 and 4.2 is the following assertion.

THEOREM 4.3. *For every $\omega > 0$ and every integer $N \geq 2$ there exists a motionless structure such that the subsidiary conditions are fulfilled for each of the structure's N bodies; that is, the structure floats freely. Moreover, it traps the mode $(\varphi_m, \mathbf{0}, \dots, \mathbf{0})$ of frequency ω ; here $\mathbf{0}$ — the zero element of \mathbb{R}^6 — is repeated N times.*

Proof. By Proposition 4.2, there exists a sufficiently large m such that $\psi_m(\nu|\mathbf{x}|, \nu y)$ has $N - 1$ local extrema on the interval $(0, j_{1,N})$ of the free surface. Locally, each of the extrema defines a family of streamlines enclosing the extrema point with their end-points on the $\nu|\mathbf{x}|$ -axis. By Proposition 4.1 (iii), there exists one more family of streamlines with $y = 0$ at both end-points of a streamline lying on each side of the singularity of ψ_m (and ϕ_m as well). See fig. 3, for examples of streamlines of both types. Choosing a single streamline from each of the described families and

complementing every of chosen streamlines in the same way as that in the middle of fig. 3 (it is clear that there are infinitely many other ways to do this), we obtain N axisymmetric bodies.

Using a properly chosen axisymmetric density distribution within each of these N bodies, we get that all subsidiary conditions concerning the bodies' equilibrium are fulfilled (see again fig. 3). Indeed, the position of the centre of mass of each body is on the y -axis and can be made arbitrarily close to the level of its lowest point. Furthermore, every matrix $\widehat{\mathbf{K}}_0^{(k)}$, $k = 1, \dots, N$, (see (6)) is a diagonal matrix with positive elements in the limit, and so $\widehat{\mathbf{K}}_0^{(k)}$ is positive definite when the centre of mass is sufficiently close to its lowest level.

It remains to show that $(\varphi_m, \mathbf{0}, \dots, \mathbf{0})$ is a trapped-mode solution of problem (7)–(10) in the case of the structure constructed above. Since φ_m satisfies relations (7) and (8) and the homogeneous condition (9) holds for it on every constructed S_k , we have to verify $6N$ equations (10) which take the following form

$$\begin{aligned} \int_{S_k} \varphi_m \partial_{\mathbf{n}} y \, ds = 0, \quad \int_{S_k} \varphi_m \partial_{\mathbf{n}} x_i \, ds = 0, \quad \int_{S_k} \varphi_m (x_2 \partial_{\mathbf{n}} x_1 - x_1 \partial_{\mathbf{n}} x_2) \, ds = 0, \\ \int_{S_k} \varphi_m \left[(y - y_0^{(k)}) \partial_{\mathbf{n}} x_i - x_i \partial_{\mathbf{n}} (y - y_0^{(k)}) \right] \, ds = 0, \quad i = 1, 2, \end{aligned} \quad (27)$$

for each of N bodies. We immediately see that the last five of these equalities are valid because every S_k is axisymmetric, φ_m depends on $|\mathbf{x}|$ and y , whereas the second factors in the integrands have the following properties:

- $\partial_{\mathbf{n}} x_i$ and $(y - y_0^{(k)}) \partial_{\mathbf{n}} x_i - x_i \partial_{\mathbf{n}} (y - y_0^{(k)})$ are odd functions of the variable x_i ;
- $x_2 \partial_{\mathbf{n}} x_1 - x_1 \partial_{\mathbf{n}} x_2$ is an odd function of both variables x_1 and x_2 .

Let us show that the first equality (27) holds for every S_k , for which purpose we apply the second Green's identity. First, we check the equality for S_N (we recall that this surface separates the singularity of φ_m from the water domain) and write the following identity

$$0 = \int_{\partial((\mathbb{R}^3 \setminus B_N) \cap C_{b,d})} (\varphi_m \partial_{\mathbf{n}} \mathcal{Y}_\nu - \mathcal{Y}_\nu \partial_{\mathbf{n}} \varphi_m) \, ds.$$

Here $\mathcal{Y}_\nu = y + \nu^{-1}$ and $C_{b,d} = \{(\mathbf{x}, y) : |\mathbf{x}| < b, -d < y < 0\}$ is a truncated cylinder and $b, d > 0$ are taken so that $S_N \subset C_{b,d}$. Since both functions in this identity are harmonic in $\mathbb{R}^3 \setminus B_N$, the boundary condition (20) yields that

$$- \int_{S_N} \varphi_m \partial_{\mathbf{n}} y \, ds = \int_{(\partial C_{b,d}) \setminus \partial \mathbb{R}_-^3} (\varphi_m \partial_{\mathbf{n}} y - \mathcal{Y}_\nu \partial_{\mathbf{n}} \varphi_m) \, ds.$$

Let us fix b and pass to the limit as $d \rightarrow +\infty$ in the last integral, which we split into the sum of two integrals: one over the bottom $\partial C_{b,d} \cap \{y = -d\}$, and the other over the lateral surface $\{|\mathbf{x}| = b, -d < y < 0\}$. The first integral tends to zero because φ_m satisfies the same estimate as R in formula (13). Therefore, we get

$$\begin{aligned} \int_{S_N} \varphi_m \partial_{\mathbf{n}} y \, ds &= 2\pi b \lim_{d \rightarrow +\infty} \int_{-d}^0 \mathcal{Y}_\nu(y) \partial_{|\mathbf{x}|} \varphi_m(\nu|\mathbf{x}|, \nu y) \Big|_{|\mathbf{x}|=b} \, dy \\ &= -2\pi \nu^{-1} \lim_{d \rightarrow +\infty} \int_{-d}^0 \mathcal{Y}_\nu(y) \partial_y \psi_m(\nu b, \nu y) \, dy. \end{aligned} \quad (28)$$

Here the axisymmetric behaviour of φ_m is taken into account in the second expression, whereas the last equality is a consequence of the first equation (21).

In order to show that the limit is equal to zero in (28), we substitute the expressions for \mathcal{Y}_ν and $\partial_y \psi_m$ into the last integral and obtain

$$-2\nu b \int_0^\infty I_1(k\nu r_m) K_1(k\nu b) \frac{k^3 dk}{k^2 + 1} \int_{-d}^0 (y + \nu^{-1}) (k \cos k\nu y + \sin k\nu y) dy \quad (29)$$

after changing the order of integration. (Indeed, the inequality $b > r_m$ and the asymptotic formulae for $I_1(z)$ and $K_1(z)$ as $z \rightarrow 0$ and $z \rightarrow +\infty$ (see, for example, [1], 9.6.7–9.6.9, 9.7.1, 9.7.2) imply that the double integral is absolutely convergent.) Since the inner integral is equal to $\nu^{-2} (1 + k^{-2}) \sin k\nu d - d\nu^{-1} (\sin k\nu d + k^{-1} \cos k\nu d)$, we get that the contribution of the first term into (29) is equal to

$$-2\nu^{-1} b \int_0^\infty k I_1(k\nu r_m) K_1(k\nu b) \sin k\nu d dk$$

tending to zero as $d \rightarrow +\infty$ by the Riemann–Lebesgue lemma. It remains to consider the contribution of the second term, which takes the form

$$\nu^{-1} \int_0^\infty \left(\sin k\nu d - k \cos k\nu d + \frac{\sin k\nu d}{\nu d} \right) f(k) dk \quad (30)$$

after integration by parts; here

$$f(k) = \left\{ I_1(k\nu r_m) K_1(k\nu b) \frac{2}{k(k^2 + 1)} + \frac{\nu r_m}{2} K_1(k\nu r) [I_0(k\nu r_m) + I_2(k\nu r_m)] - \frac{\nu b}{2} I_1(k\nu r_m) [K_0(k\nu b) + K_2(k\nu b)] \right\} \frac{k^2}{k^2 + 1}.$$

Simple analysis shows that f is integrable, and so (30) also tends to zero as $d \rightarrow +\infty$ by the Riemann–Lebesgue lemma.

To show that the first equality (27) holds for every S_k with $k \neq N$ we use the following identity

$$0 = \int_{\partial((\mathbb{R}^3 \setminus \overline{B_N \cup B_k}) \cap C_{b,d})} (\varphi_m \partial_{\mathbf{n}} \mathcal{Y}_\nu - \mathcal{Y}_\nu \partial_{\mathbf{n}} \varphi_m) ds.$$

In the same way as above, this reduces to

$$- \int_{S_N \cup S_k} \varphi_m \partial_{\mathbf{n}} y ds = - \left(\int_{S_N} + \int_{S_k} \right) = \int_{(\partial C_{b,d}) \setminus \partial \mathbb{R}^3_-} (\varphi_m \partial_{\mathbf{n}} y - \mathcal{Y}_\nu \partial_{\mathbf{n}} \varphi_m) ds.$$

Since the integral over S_N is equal to zero, the above considerations yield that the same is true for the integral over S_k with arbitrary $k \neq N$. The proof is complete.

4.4 Modified stream functions and heaving trapping structures

Let us turn to constructing a freely floating trapping structure all bodies of which are in heave motion with the same amplitude of vertical oscillations; that is,

$$\chi_*^{(1)} = \dots = \chi_*^{(N)} = \chi_H = (0, 0, 0, H, 0, 0)^\top, \quad (31)$$

where H is a (sufficiently small) positive constant. For this purpose we modify the method presented in §§ 4.2 and 4.3. Namely, we require a structure to be formed by bodies whose wetted surfaces $\{S_k\}_{k=1}^N$ are level lines of the form

$$\psi_m^{(H)}(\nu|\mathbf{x}|, \nu y) = v = \text{const}, \quad \text{where} \quad \psi_m^{(H)}(\nu|\mathbf{x}|, \nu y) = \psi_m(\nu|\mathbf{x}|, \nu y) - \frac{H}{2}(\nu|\mathbf{x}|)^2. \quad (32)$$

Indeed, relations (21) imply that

$$\partial_{\mathbf{n}}(\varphi_m - H\nu y) = 0 \quad \text{on every such } S_k,$$

which is equivalent to the Neumann condition (9) describing the heave motion. Similarly to Propositions 4.1 and 4.2, one obtains the following two propositions illustrated in fig. 4. The existence of the constants $v = v_k$ delivering trapping structures will be shown below.

PROPOSITION 4.4. *Let H be sufficiently small, then the following three assertions hold.*

- (i) *Level lines (32) are smooth curves in Q ; their end-points belong to $\partial Q \cup \{\infty\}$.*
- (ii) *A level line emanates from every point on the half-axis $\{\nu|\mathbf{x}| > 0, \nu y = 0\}$, except for the points, where $\psi_m^{(H)}(\nu|\mathbf{x}|, \nu y)$ attains its local extrema, and the point $(\nu r_m, 0)$.*
- (iii) *For every $m \geq 1$, all positive ν and all sufficiently large values of ν_N (in (32), the same notation is used as at the beginning of § 4.3) there exists a level line such that $y = 0$ at both its ends and the point $(\nu r_m, 0)$ belongs to the segment connecting these end-points.*

PROPOSITION 4.5. *For any positive integer M there exist $m_*(M)$ and $H_*(M)$ such that the stream function $\psi_m^{(H)}(\nu|\mathbf{x}|, \nu y)$ has $M - 1$ local extrema on*

$$\{\nu|\mathbf{x}| \in (0, j_{1,M}), \nu y = 0\}$$

provided $m \geq m_$ and $H \leq H_*$.*

REMARK 4.6. The level line corresponding to v and going to infinity asymptotes the vertical line $\nu|\mathbf{x}| = \sqrt{2v/H}$ (in fig. 4, such lines are located between the νy -axis and the dashed line going to infinity and to the right of the latter line).

A consequence of Propositions 4.4 and 4.5 is the following theorem.

THEOREM 4.7. *For every $\omega > 0$ and every integer $N \geq 2$ there exists a heaving structure such that the wetted surfaces of its bodies are given by level lines of $\psi_m^{(H)}$ with sufficiently small H and properly chosen values v_k , $k = 1, \dots, N$ (see fig. 4). For each of N bodies the subsidiary conditions guarantee its equilibrium; that is, the structure floats freely. Moreover, it traps the mode $(\varphi_m, \chi_H, \dots, \chi_H)$.*

Proof. Applying Propositions 4.4 and 4.5 in the same way as Propositions 4.1 and 4.2 were applied in the proof of Theorem 4.3, we obtain N axisymmetric bodies by choosing the level lines $\psi_m^{(H)}(\nu|\mathbf{x}|, \nu y) = v_k$ so that the values $\{v_k\}_{k=2}^N$ are close (but not equal) to the extrema values of $\psi_m^{(H)}(\nu|\mathbf{x}|, 0)$ on the interval $(0, j_{1,N})$; the level line $\psi_m^{(H)}(\nu|\mathbf{x}|, \nu y) = \nu_N$, where ν_N is sufficiently large, gives the wetted surface S_N of \widehat{B}_N (the cross-section of this body must be the rightmost in fig. 4).

All subsidiary conditions hold for these bodies provided axisymmetric density distributions are properly chosen within each body. As in Theorem 4.3, the velocity

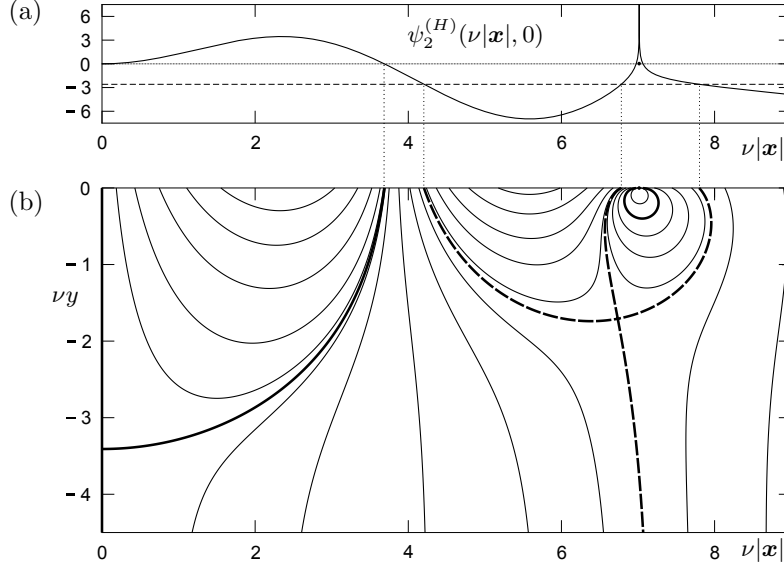


Figure 4: Let $H = 0.1$, then: the trace $\psi_2^{(H)}(\nu|\mathbf{x}|, 0)$ is plotted in (a); level lines of $\psi_2^{(H)}(\nu|\mathbf{x}|, \nu y) = v$ are plotted in (b) for various values of v . Solid bold lines correspond to $v = 0$ (nodal lines); the dashed lines in (a) and (b) correspond to $v \approx -2.590$ (the level of the right stagnation point).

potential φ_m satisfies relations (7) and (8), whereas the boundary condition (9) holds on every S_k in view of the way how these surfaces are constructed using the displacement vectors (31). It remains to verify $6N$ equations (10) of which $5N$ take the same form as equalities (27) with the exception of the first one. The latter is as follows:

$$\nu H I^{\hat{B}_k} = - \int_{S_k} \varphi_m \partial_{\mathbf{n}} y \, ds + H I^{D_k}, \quad k = 1, \dots, N. \quad (33)$$

As in the proof of Theorem 4.3, we begin with the following second Green's identity

$$0 = \int_{\partial((\mathbb{R}^3 \setminus \overline{B_N}) \cap C_{b,d})} (\varphi_m \partial_{\mathbf{n}} \mathcal{Y}_\nu - \mathcal{Y}_\nu \partial_{\mathbf{n}} \varphi_m) \, ds,$$

to which we apply the same considerations based on the boundary conditions, the behaviour of φ_m as $y \rightarrow -\infty$ and the Riemann–Lebesgue lemma with $d \rightarrow +\infty$. However, now we obtain that

$$\int_{S_N} \varphi_m \partial_{\mathbf{n}} y \, ds = H \nu \int_{S_N} (y + \nu^{-1}) \partial_{\mathbf{n}} y \, ds = -H \nu \int_{B_N} d\mathbf{x} dy + H I^{D_N}.$$

Substituting this into (33) with $k = N$, the latter equality reduces to Archimedes' law for \hat{B}_N , and so is true.

Then the same procedure yields the result for $k \neq N$, but we have to apply the second Green's identity in $(\mathbb{R}^3 \setminus \overline{B_N \cup B_k}) \cap C_{b,d}$ and to take into account the fact obtained on the previous step as well as Archimedes' law for \hat{B}_k . The proof is complete.

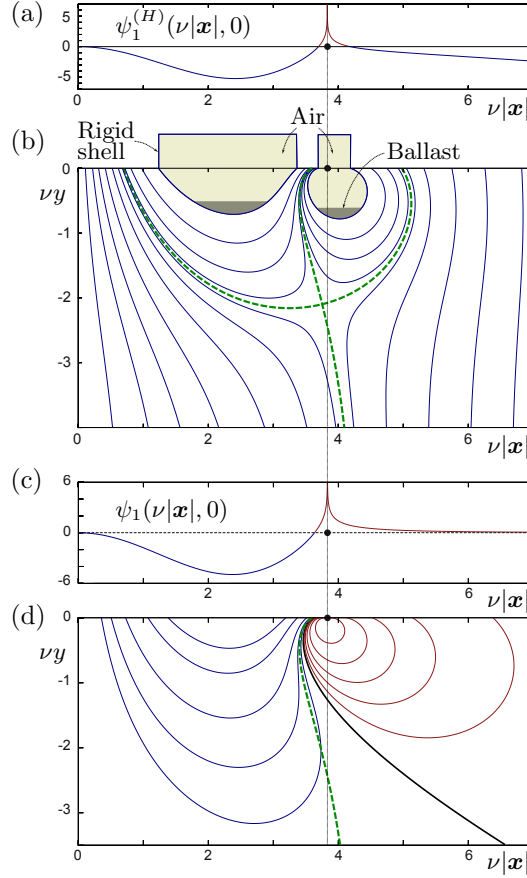


Figure 5: (a) The trace $\psi_1^{(H)}(\nu|\mathbf{x}|, 0)$ with $H = 0.1$. (b) Streamlines $\psi_1^{(H)}(\nu|\mathbf{x}|, \nu y) = v$ are plotted for $H = 0.1$ and various $v \leq 0$; the nodal line ($v = 0$) serves as S_2 (the wetted boundary of the right body); straight segments connect wetted surfaces above the free surface to form two bodies; two dashed lines ($v \approx -0.9464$) separate four different families of level lines. (c) The trace $\psi_1(\nu|\mathbf{x}|, 0)$. (d) The dashed line is taken from (b); streamlines $\psi_1(\nu|\mathbf{x}|, \nu y) = v$ are plotted for various values of v ; the nodal line ($v = 0$) is bold.

There are various ways to construct trapping structures using level lines (32), in particular, level lines plotted in fig. 4 allow us to obtain four different types of structures with $N = 2, 3$, one of which is similar to that shown in fig. 3.

4.5 Trapping structures consisting of two bodies

In the simplest case $N = 2$, we describe the whole set of trapping structures expressible in terms of ψ_1 and $\psi_1^{(H)}$ with sufficiently small H . We begin with the following proposition proved in [11], pp. 153 and 154, and illustrated in fig. 5(c).

PROPOSITION 4.8. *For any $\nu > 0$ the trace $\psi_1(\nu|\mathbf{x}|, 0)$ has the following properties. It vanishes at $\nu|\mathbf{x}| = 0$, tends to $+\infty$ as $\nu|\mathbf{x}| \rightarrow j_{1,1} \pm 0$, tends to 0 as $\nu|\mathbf{x}| \rightarrow +\infty$, has exactly one zero on $(0, j_{1,1})$, say, $\nu|\mathbf{x}|$ and exactly one extremum on $(0, j_{1,1})$, namely, the negative minimum M attained at a certain $\nu|\mathbf{x}| < \nu|\mathbf{x}|$.*

According to this proposition streamlines exist only for $v > \check{M}$ and only one streamline corresponds to each $v \in (\check{M}, 0)$ and to each $v > 0$ (see fig. 5(d)). The nodal streamline emanating from $(\nu|\hat{\mathbf{x}}|, 0)$ separates streamlines that correspond to positive and negative levels of $\psi_1(\nu|\mathbf{x}|, \nu y)$. It is straightforward to show that the nodal line does not intersect the νy -axis and goes to infinity.

Proposition 4.8 and the definition of $\psi_1^{(H)}$ yield the following corollary illustrated in fig. 5(a).

COROLLARY 4.9. *For any $\nu > 0$ and sufficiently small H the trace $\psi_1^{(H)}(\nu|\mathbf{x}|, 0)$ has the following properties. It vanishes at $\nu|\mathbf{x}| = 0$, tends to $+\infty$ as $\nu|\mathbf{x}| \rightarrow j_{1,1} \pm 0$, tends to $-\infty$ as $\nu|\mathbf{x}| \rightarrow +\infty$, has exactly one zero on $(0, j_{1,1})$, say, $\nu|\hat{\mathbf{x}}^{(H)}|$ and exactly one extremum on $(0, j_{1,1})$, namely, the negative minimum $M^{(H)}$ attained at a certain $\nu|\hat{\mathbf{x}}^{(H)}| < \nu|\hat{\mathbf{x}}^{(H)}|$.*

According to this corollary level lines such that $y = 0$ at both their end-points exist only for $v > \check{M}^{(H)}$ and only one such level line corresponds to each $v > \check{M}^{(H)}$ (see fig. 5(d)). The presence of the second negative term in the definition of $\psi_1^{(H)}$ has the following consequence. Instead of the nodal streamline separating two families of streamlines defined by ψ_1 , the same role for level lines of $\psi_1^{(H)}$ with sufficiently small H is played by the branch that corresponds to a certain critical negative level v (for $H = 0.1$ this level is ≈ -0.9464) and goes to infinity. The second branch of the critical level has $y = 0$ at its both end-points, thus separating two families of level lines having $y = 0$ at both end-points from those going to infinity (see Remark 4.6). In fig. 5(b), these two branches are shown by dashed lines.

THEOREM 4.10. *There exist four types of trapping structures defined by ψ_1 and $\psi_1^{(H)}$ with sufficiently small H . Every structure consists of two bodies and the corresponding trapped mode is either of the following four: $(\varphi_1, \mathbf{0}, \mathbf{0})$, $(\varphi_1, \boldsymbol{\chi}_H, \boldsymbol{\chi}_H)$, $(\varphi_1, \mathbf{0}, \boldsymbol{\chi}_H)$, $(\varphi_1, \boldsymbol{\chi}_H, \mathbf{0})$, where $\boldsymbol{\chi}_H$ is defined by (31).*

Proof. According to Proposition 4.8, there exist two families of streamlines defined by ψ_1 ; every streamline belonging to the first family surrounds the singularity of φ_1 , whereas streamlines of the second family are separated from the former ones by the nodal line of ψ_1 (see fig. 5(d)). Let us take S_1 and S_2 arbitrarily from different families and complement these two streamlines by, for example, rectangles in the same way as in fig. 5(b). Then we obtain a structure of two bodies satisfying all subsidiary conditions provided appropriate axisymmetric density distributions are chosen. It follows from equations (21) that the mode $(\varphi_1, \mathbf{0}, \mathbf{0})$ is trapped by this motionless structure floating freely which is guaranteed by our construction.

Considering three other cases we omit for the sake of brevity the words about complementing S_1 and S_2 by parts located above the free surface and about a proper choice of axisymmetric density distributions to satisfy all subsidiary conditions.

The same considerations as above, but using Corollary 4.9 instead of Proposition 4.8, allow us to take arbitrary level lines S_1 and S_2 from two different families defined by $\psi_1^{(H)}$ with sufficiently small H to form a structure of two bodies (see fig. 5(b)). It follows from equations (21) that the mode $(\varphi_1, \boldsymbol{\chi}_H, \boldsymbol{\chi}_H)$ is trapped by this heaving structure floating freely.

To obtain trapping structures consisting of two bodies, one of which is motionless and the other one is heaving, we use both: streamlines of ψ_1 and level lines of $\psi_1^{(H)}$. Let S_2 be an arbitrary streamline of ψ_1 surrounding the singularity. It immediately

follows from the definition of $\psi_1^{(H)}$ that the nodal line of ψ_1 lies strictly above the critical branch of $\psi_1^{(H)}$ going to infinity. Therefore, an arbitrary level line of this function can be taken as S_2 provided it lies to the left of the mentioned critical branch and has $y = 0$ at its both end-points. Again, equations (21) yield that the mode $(\varphi_1, \mathbf{0}, \chi_H)$ is trapped by this combined (motionless/heaving) structure floating freely.

Similarly, a heaving/motionless structure trapping the mode $(\varphi_1, \chi_H, \mathbf{0})$ consists of an arbitrarily taken level line of $\psi_1^{(H)}$ surrounding the singularity (it is S_2 in this case), but as S_1 we can take only any of those streamlines of ψ_1 that lie totally to the left of the critical branch of $\psi_1^{(H)}$ going to infinity. The proof is complete.

REMARK 4.11. It is possible to obtain more complicated heaving/motionless structures consisting of two bodies (see, for example, figs. 3 and 4) and also trapping structures such that each of their two bodies heaves with its own sufficiently small amplitude $H_1 \neq H_2$ (see the next section).

4.6 The general case

Here we construct general heaving/motionless structures such that they consist of N freely floating bodies and trap modes of the form

$$(\varphi_m, \chi_*^{(1)}, \dots, \chi_*^{(N)}), \quad \text{where } \chi_*^{(k)} = (0, 0, 0, H_k, 0, 0)^T, \quad k = 1, \dots, N, \quad (34)$$

and $H_k = 0$ ($H_k > 0$) corresponds to the motionless (heaving, respectively) k th body.

THEOREM 4.12. *For every integer $N \geq 2$, every $\omega > 0$ and every N -tuple (H_1, \dots, H_N) of non-negative numbers of which all positive are sufficiently small there exist a freely floating structure such that it consists of N bodies and traps the mode (34) defined by ω and (H_1, \dots, H_N) .*

Proof. The assertion is already proved in two particular cases: when all H_k are zeroes or are equal to the same sufficiently small positive number (see Theorems 4.3 and 4.7, respectively).

Since construction of a structure so that condition (9) holds on every its wetted surface S_k , $k = 1, \dots, N$, is the main point of the proof, we concentrate only on it. Indeed, there is no need to verify relations (7) and (8) because we use φ_m , whereas to show that (10) holds one has to apply the method used for this purpose in the proofs of Theorems 4.3 and 4.7. Moreover, it is always possible to choose the density distributions $\{\rho_k\}_{k=1}^N$ so that all subsidiary conditions are fulfilled for each body.

According to Propositions 4.2 and 4.5 the functions

$$\psi_m(\nu|\mathbf{x}|, \nu y) \quad \text{and} \quad \psi_m^{(H_k)}(\nu|\mathbf{x}|, \nu y)$$

have $N - 1$ local extrema on the interval $(0, j_{1,N})$ of the free surface provided m is sufficiently large and H_k is sufficiently small. Therefore, one obtains a motionless (heaving) surface S_k for every $k = 1, \dots, N - 1$ by taking a streamline (level line, respectively) corresponding to a certain value close to the extremum value of ψ_m ($\psi_m^{(H_k)}$, respectively) on the interval whose number is k counting from the origin. Chosen in this way, surfaces S_k do not overlap provided they are sufficiently small and separated by large enough spacings. Indeed, this is a consequence of the fact that the extrema of $\psi_m(\nu|\mathbf{x}|, 0)$ and $\psi_m^{(H_k)}(\nu|\mathbf{x}|, 0)$ are close to zeros of $J_0(\nu|\mathbf{x}|)$. Finally, they

must be complemented by motionless (heaving) surface S_N defined by a streamline of ψ_m (level line of $\psi_m^{(H_k)}$, respectively) corresponding to a sufficiently large positive value in order to get a small surface not overlapping with S_{N-1} . Complementing S_1, \dots, S_N by, for example, rectangles above the free surface to obtain closed shells, we complete our construction and the proof as well.

5 Conclusion

It has been shown that there exist three-dimensional structures and the corresponding time-harmonic wave modes with the following properties. Every structure consists of two or more bodies all of which have the same vertical axis of symmetry and float freely in infinitely deep water; some of these bodies (may be none) are motionless, whereas the others (may be none) heave at the frequency of the wave mode. Thus, the latter is trapped, that is, the coupled motion of the structure and water at this frequency does not radiate waves to infinity, and so, in the absence of viscosity, will persist for all time. Such structures and wave modes exist for all frequencies and the structure's geometry depends on the frequency.

References

- [1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1965.
- [2] J. T. Beale, Eigenfunction expansions for objects floating in an open sea. *Comm. Pure Appl. Math.* **30** (1977), 283–313.
- [3] D. V., Evans, R. Porter, Wave-free motions of isolated bodies and the existence of motion-trapped modes. *J. Fluid Mech.* **584** (2007), 225–234.
- [4] C. J. Fitzgerald, P. McIver, Passive trapped modes in the water-wave problem for a floating structure. *J. Fluid Mech.* **657** (2010), 456–477.
- [5] F. John, On the motion of floating bodies, I. *Comm. Pure Appl. Math.* **2** (1949), 13–57.
- [6] F. John, On the motion of floating bodies, II. *Comm. Pure Appl. Math.* **3** (1950), 45–101.
- [7] N. Kuznetsov, On uniqueness of a solution to the plane problem on interaction of surface waves with obstacle. *J. Math. Sciences* **150** (2008), 1860–1868.
- [8] N. Kuznetsov, On the problem of time-harmonic water waves in the presence of a freely-floating structure. *St. Petersburg Math. J.* **22** (2011), 985–995.
- [9] N. Kuznetsov, V. Maz'ya, B. Vainberg, *Linear Water Waves: A Mathematical Approach*. (Cambridge University Press, Cambridge 2002).
- [10] N. Kuznetsov, O. Motygin, On the coupled time-harmonic motion of water and a body freely floating in it. *J. Fluid Mech.* **679** (2011), 616–627.
- [11] N. Kuznetsov, O. Motygin, On the coupled time-harmonic motion of deep water and a freely floating body: trapped modes and uniqueness theorems. *J. Fluid Mech.* **703** (2012), 142–162.
- [12] C. M. Linton, P. McIver, Embedded trapped modes in water waves and acoustics. *Wave Motion* (2007) **45**, 16–29.
- [13] P. McIver, M. McIver, Trapped modes in the water-wave problem for a freely-floating structure. *J. Fluid Mech.* **558** (2006), 53–67.
- [14] P. McIver, M. McIver, Motion trapping structures in the three-dimensional water-wave problem. *J. Engng Math.* **58** (2007), 67–75.

- [15] C. C. Mei, M. Stiassnie, D. K.-P. Yue, *Theory and Applications of Ocean Surface Waves. Part 1: Linear Aspects*. World Scientific, 2005.
- [16] O. V. Motygin, P. McIver, On uniqueness in the problem of gravity-capillary water waves above submerged bodies. *Proc. R. Soc. A* **465** (2009), 1743–1761.
- [17] S. A. Nazarov, J. H. Videman, Trapping of water waves by freely floating structures in a channel. *Proc. R. Soc. Lond. A*. **467** (2011), 3613–3632.
- [18] P. F. Neményi, (1951) Recent developments in inverse and semi-inverse methods in the mechanics of continua. *Adv. Appl. Mech.* **2** (1951), 123–148.
- [19] J. N. Newman, Trapping of water waves by moored bodies. *J. Engng Math.* **62** (2008), 303–314.
- [20] R. Porter, D. V. Evans, Examples of trapped modes in the presence of freely floating structures. *J. Fluid Mech.* **606** (2008), 189–207.
- [21] R. Porter, D. V. Evans, Water-wave trapping by floating circular cylinders. *J. Fluid Mech.* **633** (2009), 311–325.
- [22] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *Integrals and Series, v. 2: Special Functions*. Gordon and Breach, 1986.
- [23] G. N. Watson, *A Treatise on the theory of Bessel functions*, Cambridge Univ. Press, 1944.